UDC 531.30

BIFURCATIONS AND STABILITY OF NEUTRAL SYSTEMS IN THE NEIGHBORHOOD OF THIRD ORDER RESONANCE

PMM Vol. 43, No. 3, 1979, pp. 429 –440

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(Received June 6, 1978

An autonomous system of differential equations continuously dependent on a parameter is considered. It is assumed that in the neighborhood of the bifurcation value of parameter $\mu = \mu_0$ the linear part of the system has several pairs of pure imaginary eigenvalues which at point μ_0 are related by third order resonance relation. Signs of strong stability are established and cases of bifurcation of stability properties are indicated. Interrelationship between the continuous normal form [1] and the usual normalization [2] for fixed values of the parameter is effectively applied.

1. Preliminary remarks. Let us consider the r-dimensional system of differential equations

$$z^{\cdot} = P(\mu)z + Z(z, \mu), \quad \mu \in (\mu_1, \mu_2) = D \tag{1.1}$$

continuously dependent on parameter μ, in which $P(\mu)$ is a real $r \times r$ matrix and $Z(z, \mu)$ is an r-dimensional vector function holomorphic with respect to z whose expansion in series begins with a form of order $k \geqslant 2$.

The problem of strong stability and bifurcation of stability properties in the neighborhood of the resonance value of parameter $\mu = \mu_0$ was considered in [1] on the assumption that matrix $P(\mu)$ has in region D n pairs of pure imaginary eigenvalues $\pm i\lambda_s(\mu)$ that are continuous with respect to μ.

A characteristic of point μ_0 is that at it there exist between the eigenvalues $i\lambda_s(\mu_0)$ integral linear relationships, i.e. internal resonance. It was assumed in [1] that the order of resonance is $k + 1$ with k an even number.

The obtained in [1] signs of strong stability are fairly effective when the system contains resonance of order higher than three, although these signs are also applicable to third order resonances. However cases of resonance were indicated only with resonances of orders higher than three.

Singularities related to particulars of interrelationships between the lower coefficients of the continuous and normal forms arise in investigations of systems with third order resonances [1]. Because of this, in the present investigation, which by its methods is close to that in [1], we analyze the behavior of stability properties in the neighborhood of third order resonances. One each of the two-frequency ($n = 2$) and three-frequency ($n = 3$) resonances are considered. For the remaining cases of third-order resonances the results can be modified. The exposition is conducted for the case in which the number of resonating frequencies $\pm i\lambda_s$ is the same as the system dimension.

The basic constraint is expressed as follows: matrix $P(\mu)$ has in region D the

Jordan diagonal form [3,4] with respect to μ.

2. **Two-frequency resonance of third order.** The considered system on assumptions made in Sect. 1 is represented in terms of complex conjugate variables as follows:

$$x_s^{\cdot} = i\lambda_s(\mu)x_s + X_s^{(2)}(x, \bar{x}, \mu) + X_s^{(3)}(x, \bar{x}, \mu) + \ldots, \qquad s = 1, 2 \qquad (2.1)$$

$$X_s^{(j)} = \sum_{|p|+|q|=j} a_{p,q}^{(s)}(\mu)\, x^p \bar{x}^q, \qquad p, q \in R_2^+$$

$$x^p = x_1^{p_1} x_2^{p_2}, \qquad |p| = p_1 + p_2$$

where $X_s^{(j)}$ are j-th order forms with coefficients that are continuous with respect to μ, R_n is a set of n-dimensional integral vectors, and $R_n^+ \subset R_n$ is a set of vectors with nonnegative components.

We assume that when $\mu = \mu_0$ system (2.1) has the internal resonance

$$\lambda_1^\circ + 2\lambda_2^\circ = 0, \quad \lambda_s(\mu_0) = \lambda_s^\circ \qquad (2.2)$$

where μ_0 is an isolated root of equation

$$\delta(\mu) = \lambda_1(\mu) + 2\lambda_2(\mu) = 0$$

The quantity $\delta(\mu)$ represents the resonance detuning and $\delta(\mu) \to 0$ as $\mu \to \mu_0$. The detuning $\delta(\mu)$ generally changes its sign when passing through μ_0. It will become clear later that this has a considerable effect on the stability property.

Let D be some ε-neighborhood of point μ_0. We assume ε to be so small that there are no resonances of order $\geqslant 4$ in D^*, which is a deleted at point μ_0 region D.

As in [1] we carry out continuous normalization in D and the usual normalization [2] up to third order terms in D^*.

The continuous normal form in D and the usual normal form in D^* are of the form

$$\bar{u}_s u_s^{\cdot} = i\lambda_s(\mu)\omega_s + \alpha_s(\mu)\bar{u}_1\bar{u}_2^2 + \omega_s(\alpha_{s1}\omega_1 + \alpha_{s2}\omega_2) + O_\mu(\|\omega\|^{5/2}) \qquad (2.3)$$

$$\bar{u}_s^* u_s^{*\cdot} = i\lambda_s(\mu)\omega_s^* + \omega_s^*(\alpha_{s1}^*\omega_1^* + \alpha_{s2}^*\omega_2^*) + O_\mu^*(\|\omega^*\|^{5/2}) \qquad (2.4)$$

$$\omega_s = u_s\bar{u}_s, \quad \omega_s^* = u_s^*\bar{u}_s^*$$

The nonlinearities of O_μ are continuous and bounded in D, while those of O_μ^* are continuous and unbounded in D^* as $\mu \to \mu_0$. The latter is related to that the detuning $\delta(\mu)$ appears as a small denominator in those coefficients of the normalizing transformations that are not resonance ones in the usual normalization in D^* but are resonance transformations in continuous normalization (according to definition 2.1 in [1]).

Omitting the calculations effected in the normalization process, we adduce the formulas that are required subsequently and which link the coefficients of both normal forms

$$\alpha_{11}^*(\mu) = \alpha_{11}(\mu), \quad \alpha_{12}^*(\mu) = \alpha_{12}(\mu) - 2i\delta^{-1}(\mu)\alpha_1(\mu)\bar{\alpha}_2(\mu) \qquad (2.5)$$

$$\alpha_{21}^*(\mu) = \alpha_{21}(\mu) - i\delta^{-1}(\mu)\alpha_2(\mu)\bar{\alpha}_2(\mu)$$

$$\alpha_{22}^*(\mu) = \alpha_{22}(\mu) - i\delta^{-1}(\mu)\bar{\alpha}_1(\mu)\alpha_2(\mu)$$

Note that $\alpha_s(\mu)$ are the same as the coefficients at corresponding resonance terms in input equations (2.1): $\alpha_1 = a^{(1)}_{0002}$ and $\alpha_2 = a^{(2)}_{0101}$. We do not adduce expressions for $\alpha_{js}(\mu)$ in terms of input system coefficients; formulas required for their derivation can be found in [5].

To solve the problem of strong stability and bifurcations we shall analyze the stability of system (2.1) at point μ_0 and of system (2.4) in D^*. An important part is played here by the interdependence between coefficients of both systems, as defined by formulas (2.5).

The analysis of stability of system (2.3) at point μ_0 may be based on the results obtained in [6,7] for resonances of any odd order.

Using the notation

$$A(\mu) = \operatorname{Im} \alpha_1\bar{\alpha}_2, \quad B(\mu) = \operatorname{Re} \alpha_1 \operatorname{Re} \alpha_2$$

(or $B(\mu) = \operatorname{Im} \alpha_1 \operatorname{Im} \alpha_2$ if $\operatorname{Re} \alpha_1 \operatorname{Re} \alpha_2 = 0$) in conformity with [6,7] we obtain the following statement

Theorem 2.1. System (2.3) is unstable at point μ_0, if one of the following conditions is satisfied:

a) $A(\mu_0) \neq 0$, b) $A(\mu_0) = 0$, $B(\mu_0) > 0$

and if

c) $A(\mu_0) = 0$, $B(\mu_0) < 0$

then system (2.3) is stable in the second approximation in which it has the following integral of definite sign:

$$V_0 = |\alpha_2(\mu_0)|\,\omega_1 + |\alpha_1(\mu_0)|\,\omega_2$$

(this resonance was also considered in [8,9]).

The investigation of system (2.4) in D^* can be based on Molchanov's results [10] in which an essential part is played by the numbers $\operatorname{Re} \alpha_{sj}^* = a_{sj}^*$ which in conformity with (2.5) are of the form

$$a_{11}^* = a_{11}(\mu), \quad a_{12}^* = a_{12}(\mu) + 2A(\mu)\delta^{-1}(\mu)$$
$$a_{21}^* = a_{21}(\mu), \quad a_{22}^* = a_{22}(\mu) - A(\mu)\delta^{-1}(\mu) \qquad (2.6)$$

Theorem 2.1 and (2.6) imply that the behavior of system (2.3) in D^* is closely associated with that of system (2.4) at point μ_0 and of functions $\delta(\mu)$ and $A(\mu)$ in the neighborhood of point μ_0.

In investigations of (2.4) we also use the following statement.

Theorem 2.2. For system (2.4) to be asymptotically stable at point $\mu \in D^*$ it is necessary and sufficient, in addition to the dependence on terms O_μ^*, that there exist $\gamma_1(\mu)$ and $\gamma_2(\mu) > 0$, such that the quadratic form

$$W_* = \gamma_1 a_{11}^* \omega_1^{*2} + [\gamma_1 a_{12}^* + \gamma_2 a_{21}^*]\omega_1^*\omega_2^* + \gamma_2 a_{22}^* \omega_2^{*2}$$

be negative definite in the positive cone $K = \{\omega_1^*, \omega_2^* \geqslant 0\}$.

Let us go into the proof of the theorem. First, we note that W_* is the main part of the derivative $V_*^{\cdot}$ determined on the basis of (2.4), where $V_* = \gamma_1\omega^*_1 + \gamma_2\omega^*_2$. The sufficiency is evident.

Its necessity follows from that if by the selection of $\gamma_1, \gamma_2 > 0$ it is not possible to obtain form W_* of definite negative sign, system (2.4) in the third approximation is either neutral or unstable. Specifically, when it is possible to ensure by the selection of $\gamma_1, \gamma_2 > 0$ only the negative sign constancy of W_*, then the system is neutral. If, however, the form W_* is of alternating sign for any $\gamma_1, \gamma_2 > 0$, stability of the system in the third approximation is implied by the existence in it (in that approximation) of unstable rays [10]. Presence of the latter guarantees instability of the whole system.

As a corollary of the results of investigation of system (2.4) close to resonance, we obtain the following statement. (Continuity of all functions dependent on μ in D or D^*, as well as a reasonable smallness of region D are assumed and used throughout the subsequent analysis).

T h e o r e m 2 . 3. Let $A(\mu_0)$ and $\sigma(\mu_0) = a_{11}(\mu_0) + 2a_{21}(\mu_0) \neq 0$. Then for the asymptotic stability of system (2.4) at point $\mu \in D^*$, independently of terms $O_\mu{}^*$, it is necessary and sufficient that the following conditions are satisfied:

$$1)\ a_{11}(\mu) < 0, \quad 2)\ A(\mu)\delta^{-1}(\mu) > 0, \quad 3)\ \sigma(\mu) < 0$$

P r o o f. Smallness of region D and the limit relationship $\delta(\mu) \to 0$ as $\mu \to \mu_0$ together with condition $A(\mu_0) \neq 0$ guarantee in accordance with (2.6) the fulfilment of equalities

$$\operatorname{sign} a_{12}{}^*(\mu) = -\operatorname{sign} a_{22}{}^*(\mu) = \operatorname{sign} A(\mu)\delta^{-1}(\mu) \quad (\forall \mu \in D^*) \tag{2.7}$$

For form W_* to be of negative definite sign for $\gamma_1, \gamma_2 > 0$ it is necessary that $a_{11}{}^*(\mu)$ and $a_{22}{}^*(\mu) < 0$. Taking into account (2.6) and (2.7) we obtain the necessity of conditions 1 and 2 of the theorem.

Assuming that conditions 1 and 2 are satisfied, we can clarify when there exists the necessary selection of γ_1 and $\gamma_2 > 0$. It follows from (2.7) that $a_{12}{}^*(\mu) > 0$ ($\forall\ \mu \in D^*$). If $a_{21} < 0$, the necessary selection of γ_1 and $\gamma_2 > 0$ evidently exists and automatically $\sigma(\mu) < 0$. If $a_{21} \geqslant 0$, the selection of $\gamma_1, \gamma_2 > 0$ exists then and only then when

$$a_{11}a_{22} - a_{12}a_{21} - \sigma(\mu)A(\mu)\delta^{-1}(\mu) > 0$$

The coefficients a_{js} are bounded in D and function $\sigma A\delta^{-1}$ by the theorem conditions is unbounded in D^*. But then the last inequality is equivalent to condition $\sigma(\mu) < 0$.

Thus the required selection of $\gamma_1, \gamma_2 > 0$ which satisfies Theorem 2.2. exists only when conditions 1 — 3 are satisfied.

R e m a r k. The proof shows that the stipulation $A(\mu_0)$ and $\sigma(\mu_0) \neq 0$ of Theorem 2.3 can be relaxed by the substitution for them of the requirement of unboundedness of function $\sigma(\mu)A(\mu)\cdot\delta^{-1}(\mu)$. That condition is clearly satisfied in the input assumptions. It can also be satisfied when both or one of functions $A(\mu)$ and $\sigma(\mu)$ vanish at point $\mu = \mu_0$.

We pass to the investigation of behavior of the stability property when parameter μ is varied.

T h e g e n e r a l c a s e. We assume that

$$a_{11}(\mu_0) \neq 0, \quad \sigma(\mu_0) \neq 0, \quad A(\mu_0) \neq 0 \tag{2.8}$$

When the first two inequalities in (2.8) are satisfied, it is sufficient for the verification of conditions 1 and 3 of Theorem 2.3. at point $\mu \in D^*$ to check their validity at point μ_0. If, furthermore $A(\mu_0) \neq 0$, the test of condition 2 reduces to the determination of the sign of function $\delta(\mu)$ in the neighborhood of its isolated zero μ_0. In connection with this we introduce the sets $D_{\pm}^* = \{\mu \mid A(\mu_0)\delta(\mu) \gtrless 0\}$. When one of these is empty, the second is the same as D^*. (This corresponds to the case when δ does not change its sign when passing through μ_0).

Let conditions 1 and 3 be satisfied at point μ_0. Then, by virtue of continuity they are satisfied in region D. Since condition 2 is satisfied on set D_+^*, Theorem 2 makes it possible to assert that system (2.4) is asymptotically stable for any $\mu \in D_+^*$. It is on this set that all three conditions of the theorem are satisfied.

Let us investigate the behavior of system (2.4) in the considered general case,when even one of conditions of Theorem 2.3 is violated at point μ_0. For this we shall elucidate in conformity with [10] whether the system can be neutral in the third approximation.

It follows from (2.6) and (2.8) that $a_{11}^*(\mu)$, $a_{22}^*(\mu) \neq 0$ and $\forall \mu \in D^*$. Hence system (2.4) has no neutral lines on the one-dimensional faces of cone K for all $\mu \in D^*$.

Existence of neutral lines inside cone K depends on the determinant

$$\Delta^* = \begin{vmatrix} a_{11}^* & a_{12}^* \\ a_{21}^* & a_{22}^* \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21} - \sigma(\mu) A(\mu) \delta^{-1}(\mu)$$

being zero.

The first two of inequalities (2.8) imply that $\Delta^* \neq 0$ and $\forall \mu \in D^*$.

Thus, under conditions of the general case system (2.4) is unstable in the third approximation if even a single condition of Theorem 2.3 is violated. It can be shown that the instability is coarse (is independent of nonlinearities of O_μ^*).

The above can be summarized by the following theorem.

Theorem 2.4. Let relationships (2.8) be satisfied. Then, if inequalities 1 and 3 of Theorem 2.3 are satisfied at point $\mu = \mu_0$, system (2.4) is asymptotically stable on set D_+^* and unstable on D_-^*. When condition 1 or 3 is violated at point μ_0, the system is unstable for all $\mu \in D^*$.

Theorems 2.1 – 2.4 make possible the complete analysis of stability properties in the considered general case.

Results of the above analysis are tabulated below, where the following symbols are used: AS for asymptotic stability, I for instability, B for bifurcation, and SI for strong instability.

The Table shows that two types of bifurcation may appear in system (2.1) ; an explosive instability characterized by asymptotic stability in D^* and instability at point μ_0, and a one-sided explosive instability when the asymptotic stability in D_+^* changes at point μ_0 to instability which is retained in D_-^*. (Note that when $A(\mu_0) \neq 0$ the sets $D_{\pm}^*$, if both are not empty, coincide with one of the half-neighborhoods of point μ_0.)

Table

$a_{11}(\mu_0)$	$\sigma(\mu_0)$	PROPERTIES OF STABILITY			
		D_-^*	μ_0	D_+^*	D
$-$	$-$	I	I	AS	B
$-$	$+$	I	I	I	SI
$+$	$\pm$	I	I	I	SI
$-$	$-$	$\varnothing$	I	AS $D_+^* = D^*$	B
$-$	$+$	$\varnothing$	I	I $D_+^* = D^*$	SI
$\pm$	$\pm$	I $D_-^* = D^*$	I	$\varnothing$	SI

The instability at point μ_0 when $A(\mu_0) \neq 0$ explains the absence of strong asymptotic stability in the Table.

Degenerate cases. Let us consider the cases of violation of conditions (2.8).

First, we note that Theorems 2.1 –2.4 make possible the analysis of stability in D also when the first two of conditions (2.8) are violated. To do this it is necessary to obtain information on the behavior of functions $a_{11}(\mu)$ and $\sigma(\mu)$ in the neighborhood of point μ_0 which now is the zero of at least one of these functions.

Let us go into the case when $A(\mu_0) = 0$ on the assumption that the first two of conditions (2.8) are satisfied. The behavior of system (2.1) at point μ_0 corresponds now to cases b) and c) in Theorem 2.1.; in case c) it is necessary to resort in the analysis to terms of higher order. To obtain sufficient signs of stability we use the integral of the model system V_0 as the Liapunov function.

Investigation of the system in D^* substantially depends on the behavior of function $A(\mu)\delta^{-1}(\mu)$ in D^*. On the assumption that $A(\mu) \neq 0$ and $\forall \mu \in D^*$ we may have two cases:

α) function $A(\mu)\delta^{-1}(\mu)$ is unbounded in D^*,

β) function $A(\mu)\delta^{-1}(\mu)$ is bounded in D^*.

Investigation of system (2.4) in D^* in case α, as well as in the general case can be carried out using Theorem (2.3) (see the Remark). Taking into account that $A(\mu_0) = 0$ we obtain the following definition of sets $D_\pm^*$: $D_\pm^* = \{\mu \mid A(\mu)\delta(\mu) \gtrless 0\}$. As previously, when conditions 1 –3 of Theorem 2.3 are satisfied at point μ_0 system (2.4) is asymptotically stable on set D_+^* and unstable on set D_-^*.

In case β we assume that

$$\beta') \quad \exists \lim_{\mu \to \mu_0} A(\mu)\delta^{-1}(\mu) = k < \infty$$

and additionally

$$a_{22}(\mu_0) \neq k, \quad a_{12}(\mu_0) \neq -2k \tag{2.9}$$

$$\Delta(\mu_0) = a_{11}(\mu_0)a_{22}(\mu_0) - a_{21}(\mu_0)a_{12}(\mu_0) \neq k\sigma(\mu_0)$$

The application of Theorem 2.2 shows that under conditions β' and (2.9) for the asymptotic stability of system (2.4) at point μ it is necessary and sufficient if one of the following groups of conditions are satisfied:

$$a_{11}(\mu) < 0, \quad a_{22}(\mu) < k; \quad a_{12}(\mu) + 2k < 0 \quad \text{or} \quad a_{21}(\mu) < 0 \tag{2.10}$$

$$a_{11}(\mu) < 0, \quad a_{22}(\mu) < k; \quad \Delta(\mu) > k\sigma(\mu) \tag{2.11}$$

The first two of conditions (2.8) and conditions (2.9) make it possible to conclude that in the considered case system (2.4) cannot be neutral. Hence the violation of conditions (2.10) and (2.11) at some point $\mu \in D^*$ results in instability of system (2.4) at that point. Assumption (2.9) makes it possible to restrict the test of conditions (2.10) and (2.11) to $\mu = \mu_0$. Hence the violation of conditions (2.10) and (2.11) at point μ_0 results in instability in D^*.

The above exposition and Theorem 2.1 lead to the following conclusion. Let $B(\mu_0) > 0$. If conditions (2.10) or (2.11) are satisfied at point μ_0, then we have at that point a bifurcation of the type of explosive instability. In other cases we have a strong instability.

When $A(\mu_0) = 0$, $B(\mu_0) < 0$ the behavior of the system in D^* is exactly the same as described above. Asymptotic stability is possible at point μ_0 for which it is sufficient for the form

$$|\alpha_2| a_{11}\omega_1^2 + (|\alpha_2| a_{12} + |\alpha_1| a_{21})\omega_1\omega_2 + |\alpha_1| a_{22}\omega_2^2$$

to be of definite (negative) sign at point μ_0 with ω_1 and $\omega_2 \geqslant 0$.

Because of this, when this form is definite negative and conditions (2.10) or (2.11) are satisfied at point μ_0 we have a strong asymptotic stability.

Let now $A(\mu) \equiv 0$ $(\forall \mu \in D)$. The behavior of the stability property of system (2.4) in D^* in the third approximation, as implied by (2.6), is independent of stability properties of the system at the resonance point μ_0. However in the case of asymptotic stability of (2.4), the closeness to resonance apparently affects the size of the attraction region, since as previously $O_\mu^* \to \infty$, as $\mu \to \mu_0$. The necessary and sufficient conditions of asymptotic stability are defined in Theorem 2.2, and the behavior of system (2.3) at point μ_0 was described above.

Note that in practically important situations the case of $A(\mu) \equiv 0$ may be accompanied by the identities $a_{sj}(\mu) \equiv 0$. Investigations in D^* can be based on the results in [11, 12].

The situation described above is obviously realized in Hamiltonian systems, for which this problem can be solved on the basis of a number of well known results, in the theory of Hamiltonian systems using the continuous normal form.

We would also point out that the case when $A(\mu_0) = 0$ but $A(\mu) \neq 0$ $(\forall \mu \in D^*)$ may occur in the analysis of systems that are close to Hamiltonian.

3. The three-frequency resonance. Let us consider system (2.1) in which $s = 1, 2, 3$, and assume that it has a unique internal resonance of the third order

$$\lambda_1(\mu_0) + \lambda_2(\mu_0) + \lambda_3(\mu_0) = 0 \tag{3.1}$$

The normalization process in D and D^* yields the following systems:

$$\bar{u}_s u_s^{\cdot} = i\lambda_s(\mu)\,\omega_s + \alpha_s\bar{u}_1\bar{u}_2\bar{u}_3 + \omega_s\Sigma\alpha_{sj}\omega_j + O_\mu(\|\omega\|^{5/2}) \tag{3.2}$$

$$\bar{u}_s^* u_s^{*\cdot} = i\lambda_s(\mu)\,\omega_s^* + \omega_s^*\Sigma\alpha_{sj}^*\omega^*_j + O_\mu^*(\|\omega^*\|^{3/2}) \tag{3.3}$$

that correspond to the continuous normal form in D and the usual normal form in D^*

The properties of nonlinearity of O_μ, and O_μ^* are analogous to those described in Sect. 2. The relation between coefficients of normal forms is

$$\alpha_{sj}^* = \alpha_{sj} - i\alpha_s\bar{\alpha}_k\delta^{-1}(\mu), \quad j \neq s \neq k; \quad j,\, s,\, k = 1,\, 2,\, 3 \tag{3.4}$$

$$\alpha_{jj}^* = \alpha_{jj}, \quad \delta(\mu) = \lambda_1(\mu) + \lambda_2(\mu) + \lambda_3(\mu) \to 0 \quad \text{for} \quad \mu \to \mu_0$$

We introduce the notation $a_{sj}(\mu) = \operatorname{Re}\alpha_{sj}(\mu)$ and $c_{sj} = \operatorname{Im}\alpha_s(\mu)\bar{\alpha}_k(\mu)$ (note that $c_{sj} = -c_{kj}$).

The general case. We assume that at point μ_0

$$c_{sj}(\mu_0) = c_{sj}^\circ \neq 0 \quad (\forall s, j \mid s \neq j), \quad a_{jj}(\mu_0) \neq 0 \tag{3.5}$$

We denote $c_{13}^\circ = \alpha$, $c_{21}^\circ = \beta$, and $c_{12}^\circ = \gamma$.

Investigation of system (3.1) with conditions (3.5) and $\mu = \mu_0$ is based on the following results obtained in [6, 13].

Theorem 3.1. Let conditions (3.5) be satisfied. Then system (3.2) is stable at point μ_0 in the second approximation then and only then when

$$\operatorname{sign}\alpha = \operatorname{sign}\beta = -\operatorname{sign}\gamma \tag{3.6}$$

(which corresponds to two combinations of signs $(++-)$ and $(--+)$). In other cases the system is unstable irrespective of the linearity of O_μ.
When (3.6) is satisfied, system (3.2) in the second approximation at point μ_0 admits the following integral of definite sign:

$$V_0 = \beta\omega_1 - \gamma\omega_2 + \alpha\omega_3 \tag{3.7}$$

The theorem thus formulated is a corollary of Theorems 2.1, 2.2 and (3.1) in [6]. Conditions (3.5) and (3.6) ensure that conditions (2.2) and (2.5) A in [6] are satisfied. The form of the integral (3.7) is established by direct calculation, as in [13].

Let us now consider system (3.3) in D^*. Its analysis in the third approximation conforms to [10]. We pass in (3.3) to variables ω_s^* and consider the model system

$$\tfrac{1}{2}\omega_s^{*\cdot} = \omega_s^*\Sigma a_{si}^*\omega_j^* \tag{3.8}$$

$$(a_{sj}^* = \operatorname{Re}\alpha_{sj}^* = a_{sj}(\mu) + c_{sj}(\mu)\,\delta^{-1}(\mu),\ s \neq j;\ a_{jj}^* = a_{jj}(\mu))$$

where all coefficients a_{sj} and c_{sj} are continuous and bounded in D.

We introduce matrix $A = (a_{sj}^*)_1^3$ and A_j which are obtained from A by cancelling the j-th column and j-th row. We then consider the totality of seven systems of equations

$$Aq = l, \quad A_j q^{(j)} = l^{(j)}, \quad j = 1, 2, 3; \quad a_{ss}q_s = 1, \quad s = 1, 2, 3 \tag{3.9}$$

where $q = (q_1, q_2, q_3)$, $l = (l_1, l_2, l_3)$, and $l_j = 1$ or $0\ (\forall_j)$, and $q^{(j)}$ and $l^{(j)}$ are projections of vectors q and l on the planes $q_j = 0$ and $l_j = 0$.

It was shown in [10] that for system (3.8) to be unstable it is necessary and sufficient that the totality [of systems] (3.9) has a strictly positive solution for $l \neq 0$.

Instability is ensured by the existence of an increasing particular solution contained either inside the cone $K = \{\omega_1^*, \omega_2^*, \omega_3^* \geqslant 0\}$ when the three-dimensional system in (3.9) has a positive solution, or on the one- or two-dimensional faces of cone K, if one of the corresponding systems of the totality (3.9) has a positive solution.

It is obviously clear that when

$$(\exists j) \quad (a_{jj}(\mu_0) > 0) \tag{3.10}$$

then system (3.8) is unstable, since there exists an increasing solution on the j-th one-dimensional face of cone K.

Let now

$$(\forall j)(a_{jj}(\mu_0) < 0) \tag{3.11}$$

Let us ascertain if there exists an increasing solution on the two-dimensional faces of cone K.

The calculation of determinants $\Delta_j = |A_j|$ yields

$$\Delta_1 = -\alpha\gamma\delta^{-2} + O(\delta^{-1}), \quad \Delta_2 = \alpha\beta\delta^{-2} + O(\delta^{-1})$$
$$\Delta_3 = -\beta\gamma\delta^{-2} + O(\delta^{-1})$$

These equalities show that the determinants Δ_j under conditions (3.5) are nonzero and retain their sign in D^*. From the analysis of system $A_j q^{(j)} = l^{(j)}$ with $l^{(j)} = (1; 1)$ we conclude that at least one of these has a positive solution when conditions (3.6) are violated.

Thus the coarse instability of system (3.2) at point μ_0 is linked to the violation of (3.6) and results in instability also in D^*.

If conditions (3.6) are satisfied, system (3.8) has no increasing solutions on the two-dimensional faces. Instability of the system is only possible in the presence of an increasing solution inside the cone K.

For the investigation of system $Aq = l \neq 0$ we, first of all, determine its determinant

$$\Delta = |A| = \Delta_0\delta^{-2} + O(\delta^{-1})$$
$$\left(\Delta_0 = -\alpha\gamma \sum_{j=1}^{3} a_{j1} + \alpha\beta \sum_{j=1}^{3} a_{j2} - \beta\gamma \sum_{j=1}^{3} a_{j3}\right)$$

We assume that $\Delta_0 \neq 0$. Then the determinant $\Delta \neq 0$ and retains its sign in D^*. Calculations show that an increasing solution exists inside cone K, if condition $\Delta_0 > 0$ is satisfied. Since all determinants $\Delta_s \neq 0$ (by virtue of (3.5)), there are no neutral lines on the faces of cone K. They are not present on the one-dimensional faces when $a_{jj}(\mu_0) \neq 0\ (\forall j)$, and inside cone K when $\Delta_0 \neq 0$. Hence when conditions (3.6) and

$$\Delta_0(\mu_0) < 0, \quad a_{jj}(\mu_0) < 0 \quad (\forall j) \tag{3.12}$$

are satisfied, system (3.8) is asymptotically stable in D^*.

When conditions (3.6) are satisfied, only the stability of the initial system at point μ_0 remains uninvestigated. To obtain sufficient signs of stability we use the integral of the model system (3.7). Its derivative calculated for $\mu = \mu_0$ on the basis of (3.2) is

$$V_0^{\cdot} = \beta a_{11}^{\circ}\omega_1^2 + (\beta a_{12}^{\circ} - \gamma a_{21}^{\circ})\omega_1\omega_2 + (\beta a_{13}^{\circ} + \alpha a_{31}^{\circ})\omega_1\omega_3 - \gamma a_{22}^{\circ}\omega_2^2 + (-\gamma a_{23}^{\circ} + \alpha a_{32}^{\circ})\omega_2\omega_3 + \alpha a_{33}^{\circ}\omega_3^2 + O(\|\omega\|^{5/2}) = W_2 + O(\|\omega\|^{5/2})$$

If the form W_2 is of definite sign in cone K, its sign when $a_{jj}^{\circ} < 0$ is opposite to that of V_0, and this leads to sufficient conditions for asymptotic stability. It remains to check whether the requirement for form W_2 to be of definite sign when $a_{jj}^{\circ} < 0$ contradicts conditions (3.6).

For this we set in W_2 $\omega_1 = -\alpha\gamma$, $\omega_2 = \alpha\beta$, and $\omega_3 = -\beta\gamma$. All these figures in conditions (3.6) are positive. At that point of cone K we have for W_2 the expression

$$W_2 = -\alpha\beta\gamma\Delta_0$$

hence $\operatorname{sign} W_2 = \operatorname{sign} \Delta_0$.

Thus the necessary conditions for form W_2 to be of positive sign opposite to that of V_0 are

(3.13) The pattern of signs in (3.6) must be $(+ + -)$ and $\Delta_0 < 0$,

(3.14) The pattern of signs in (3.6) must be $(- - +)$ and $\Delta_0 > 0$.

The existence of forms W_2 of definite sign opposite to that of V_0 when conditions (3.13) and (3.14) are satisfied is fairly obvious. Thus, for example, if in W_2 all $a_{js}^{\circ} < 0$ and the pattern of signs is $(+ + -)$, then (3.13) is satisfied, and W_2 has the required property of definite sign. If among a_{jj}° there are positive ones, it is not difficult to ascertain that system (3.2) with conditions (3.5) satisfied is unstable in the third approximation. (It can be shown that the instability is coarse).

From the above follows the theorem.

T h e o r e m 3 . 2. Let the investigated system with resonance (3.1) be such that conditions (3.5) and $\Delta_0(\mu_0) \neq 0$ are satisfied.

1) If either (3.10) is satisfied or (3.6) is violated, there is a considerable instability at point μ_0.

2) If (3.10) is violated (i.e. (3.11) holds) and (3.6) is satisfied, then, when $\Delta_0(\mu_0) > 0$, the system is unstable in D^*. If then the pattern of signs in (3.6) is $(- - +)$ and form W_2 is of definite sign, we have bifurcation at point μ_0, which shows that instability in D^* is replaced by asymptotic stability at point μ_0.

3) If (3.10) is violated and (3.6) satisfied, the system is asymptotically stable in D^*, provided that $\Delta_0 < 0$. If the pattern of signs in (3.6) is $(+ + -)$ and the form W_2 is of definite sign in cone $K = \{\omega_j \geqslant 0\}$, we have at point μ_0 strong asymptotic stability.

This theorem does not include the case of $\Delta_0(\mu_0) = 0$. That case can be analyzed using the properties of $\Delta_0(\mu)$ in D^*. We shall not dwell on the investigation of cases associated with vanishing of some of the numbers $a_{jj}(\mu_0)$ and $c_{sj}(\mu_0)$. An example of the analysis of degenerate cases was given in Sect. 2 for two-frequency resonance.

We note in conclusion that the application of the majority of results obtained in Sects. 2 and 3 requires the knowledge of coefficients of the continuous normal form only at the resonance point μ_0, supplemented by the knowledge of resonance detuning $\delta(\mu)$ in the neighborhood of point μ_0. Hence it is sufficient in practice to derive the normal form only at point $\mu = \mu_0$ for solving the problem of strong stability and bifurcations.

The situation in which the sign of the detuning $\delta(\mu)$ is unknown is fairly real. It is obviously realized when the eigenvalues of all parameters are only approximately known. On the basis of this investigation it is possible to state that the analysis of stability under such conditions is far from reliable.

Considering the approximate system as imbedded in set (1.1), so that some unknown value of parameter $\mu^* \in D$ corresponds to it in that set. Investigation of stability yields reliable results, if system (1.1) does not contain bifurfaction of the stability property and, thus, the character of stability is the same for all $\mu \in D$, including μ^*.

The test for the presence of bifurcation does not require the knowledge of the detuning sign and of the exact values of coefficients of the normal form, only the signs of some coefficients of the continuous normal form are needed, and their approximate values can be used.

The investigation of strong stability of system parametrically disturbed may thus be, considered as an investigation of stability of systems known only approximately.

4. **E x a m p l e.** Let us consider the system of differential equations

$$z_s^{\cdot\cdot} + \lambda_s^2(\mu) z_s = Z_s^{(2)}(\mu, z, z^\cdot) + Z_s^{(3)}(\mu, z, z^\cdot) + \ldots + = Z_s(\mu, z, z^\cdot) \quad (4.1)$$

restricting the investigation to possible bifurcations in this system.

We write the forms $Z_s^{(2)}$ and $Z_s^{(3)}$ in the form

$$Z_s^{(2)} = \sum_{j,\, h=1}^{n} a_{jh}^{(s)} z_j z_h + b_{jh}^{(s)} z_j z_h^\cdot + c_{jh}^{(s)} z_j^\cdot z_h^\cdot \quad (4.2)$$

$$Z_s^{(3)} = \sum_{j,\, h,\, k=1}^{n} a_{jhk}^{(s)} z_j z_h z_k + b_{jhk}^{(s)} z_j z_h z_k^\cdot + c_{jhk}^{(s)} z_j z_h^\cdot z_k^\cdot + d_{jhk}^{(s)} z_j^\cdot z_h^\cdot z_k^\cdot$$

By substituting variables $x_s = z_s - i\lambda_s^{-1} z_s^\cdot$ we reduce (4.1) to the system

$$x_s^\cdot = i\lambda_s x_s + X_s^{(2)}(x, \bar{x}, \mu) + X_s^{(3)}(x, \bar{x}, \mu) + \ldots \quad (4.3)$$

Let us investigate the structure of coefficient of forms $X_s^{(2)}$ and $X_s^{(3)}$. It can be established (see [5]) that the real parts of these coefficients are made up using only a combination of coefficients $b_{jh}^{(s)}$ ($b_{jhk}^{(s)}$, $d_{jhk}^{(s)}$), and their imaginary parts by using $a_{jh}^{(s)}$, $c_{jh}^{(s)}$ ($a_{jhk}^{(s)}$, $c_{jhk}^{(s)}$).

In conformity with this we divide all terms in (4.2) in two groups. To the first group we assign terms with coefficients $b_{jh}^{(s)}$, $b_{jhk}^{(s)}$, and $d_{jkh}^{(s)}$, and the remaining to the second.

If in (4.1) there are only terms of the first group, the coefficients for forms $X_s^{(2)}$ and $X_s^{(3)}$ are pure imaginary. It can be shown that then the coefficients in both normal forms are pure imaginary in the second and third orders.

If, however, (4.1) contains only terms of the second group, these coefficients in (4.3) are real. In the continuous normal form the coefficients at second order terms are real. In the third order the coefficients in both normal forms are complex but, when all $b_{jhk}^{(s)} = d_{jhk}^{(s)} = 0$, they are pure imaginary.

Let system (4.1) with $s = 1, 2$ have resonance (2.2). It follows from the above that when only terms of the first group are present in (4.1), the coefficients α_1, α_2, α_{js}, and α_{js}^* are pure imaginary. But then $A(\mu) \equiv 0$ and $a_{js} \equiv 0$, and conditions (2.7) and (2.8) are violated.

If only terms of the second group appear in (4.1), then α_1 and α_2 are real, $A(\mu) \equiv 0$, and (2.8) is violated. When coefficients α_{js} are complex, condition (2.7) is generally satisfied. It is obviously not satisfied when $b_{jhk}^{(s)} = d_{jhk}^{(s)} = 0$.

Both cases are degenerate and were discussed at the end of Sect. 2. Note that in both of them $B(\mu) \neq 0$ and, when $B(\mu_0) > 0$, then in both degenerate cases the following types of bifurcation may be realized in system (4.1). Instability changes at point μ_0 to stability in any finite order in D^*. For the realization of such case it is sufficient, for instance, that function $Z_s(\mu, z, z^*)$ be independent of z^* or contained z^* only of even powers. When these conditions are satisfied, the coefficients of both normal forms are pure imaginary in any order.

If both groups of terms are present in (4.1), the coefficients of the two normal forms are complex, and all cases described in the course of investigation of the general case in Sect. 2 can occur.

Let now in (4.1) $s = 1, 2, 3$ and (3.1) be satisfied. As in the case of two-frequency resonance, the general case can only be realized when (4.1) contains terms of both groups, and then Theorem 3.2 applies. If only terms of the first group are present in (4.1), all of formulas (3.5) are invalid, since $c_{sj} = \operatorname{Im} \alpha_s \bar{\alpha}_k \equiv 0$ and $a_{sj} = \operatorname{Re} \alpha_{sj} \equiv 0$. If (4.1) contains only terms of the second group, then again $c_{sj} \equiv 0$, but a_{jj} can be nonzero.

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Translated by J. J. D.